Billiard language of polygons

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Consider a polygonal billiard: given a starting unit tangent vector on the polygon, flow along it, and each time the path hits an edge, reflect the tangent vector about that edge; stop only if hitting a vertex. To each path we can associate a word consisting of all the edges and vertices hit by the path, going forward and backward. Call diagonal a path that starts at a vertex and ends at a vertex. A path has a geometric length and a combinatorial length. (Or restrict to paths that never hit a vertex, and use only edges as the alphabet.)

The language of a path is the set of factors of its word. The language of the billiard is the union of languages of its paths. The complexity of a language is the function which to each integer associates the number of words of that length in the language.
Musical version: use musical notes as labels; then each path has a melody.

How are a polygon and its billiard language related?

- Duchin-Erlandsson-Leininger-Sadanand prove the language of a polygonal billiard determines it (up to some geometric transformations).
- Enumerate diagonal words, periodic words? their combinatorial lengths?
- Asymptotics of the number of diagonals of bounded combinatorial length?
- Recognize billiard words? their factors?
- Eventual complexity of aperiodic words? Hubert gives it for rational polygons.
- Language complexity? Cassaigne-Hubert-Troubetzkoy give asymptotics for polygons that two-color-reflection-tile the plane.
- Lengths and non-lengths of periods of periodic words?

In a regular polygon, one can refine some of the questions by asking them about paths with specified symmetries.

Davis-Lelièvre give a guess for the complexity asymptotics for the regular pentagon.

References

- Hubert, Bull Soc Math Fr 1995
- Cassaigne-Hubert-Troubetzkoy, arXiv:math/0109208, Ann Inst Fourier 2002
- Duchin-Erlandsson-Leininger-Sadanand, arXiv:1804.05690, Comm Math Helv 2021+
- Davis-Lelièvre, arXiv:1810.11310, Journal Modern Dynamics 2022+
